CYK Tensors, Maxwell Field and Conserved Quantities for Spin-2 Field

Jacek Jezierski*

Department of Mathematical Methods in Physics,
University of Warsaw, ul. Hoża 74, 00-682 Warsaw, Poland

Abstract

Starting from an important application of Conformal Yano–Killing tensors for the existence of global charges in gravity (which has been performed in [17] and [18]), some new observations at $\mathscr{I}^+$ are given. They allow to define asymptotic charges (at future null infinity) in terms of the Weyl tensor together with their fluxes through $\mathscr{I}^+$. It occurs that some of them play a role of obstructions for the existence of angular momentum. Moreover, new relations between solutions of the Maxwell equations and the spin-2 field are given. They are used in the construction of new conserved quantities which are quadratic in terms of the Weyl tensor. The obtained formulae are similar to the functionals obtained from the Bel–Robinson tensor.

1 Introduction

The global charges result in a natural way from a geometric formulation of the "Gauss law" for the gravitational charges, defined in terms of the Riemann tensor. They lead to the notion of the *Conformal Yano–Killing tensor* (see [17] and [18]). A Conformal Yano–Killing (CYK) equation (3.3) possesses twenty-dimensional space of solutions for flat Minkowski metric in four-dimensional spacetime. There is no obvious correspondence between ten-dimensional asymptotic Poincaré group and the twenty-dimensional space of CYK tensors. Only half of them (the four-momentum vector p_μ and the angular momentum tensor $j_{\mu\nu}$) are Poincaré generators. This situation is analogous to that of electrodynamics, where, in topologically nontrivial regions, we have two charges (electric + magnetic) despite the fact that the gauge group is only one-dimensional. Let us notice, that for $n = 2$ (n is the dimension of spacetime) the space of solutions of the equation (3.3) is infinite and for $n = 3$ the corresponding space is only four-dimensional. Possible dimensions are summarized below:

dimension of spacetime	$n = 2$	$n = 3$	$n = 4$
dimension of (pseudo)euclidean group	3	6	10
dimension of conformal group	∞	10	15
dimension of space of CYK tensors	∞	4	20

The above table shows that there is no obvious relation between CYK tensors and the group. In the most interesting case $n = 4$ CYK tensors are related to eleven-dimensional group of Poincaré transformations enlarged by dilatation (pseudo-similarity transformations). Eleven-dimensional algebra of this group allows us to construct (via the wedge product) all CYK tensors in Minkowski spacetime.

---

*Partially supported by Swiss National Fonds. E-mail: Jacek.Jezierski@fuw.edu.pl

A natural application of the CYK tensors to the description of asymptotically flat spacetimes was proposed in [17]. It allows us to define an asymptotic charge at spatial infinity without supertranslation ambiguities. The existence or nonexistence of the corresponding asymptotic CYK tensors can be chosen as a criterion for the classification of asymptotically flat spacetimes at spatial infinity. A definition of a *strong* asymptotic flatness, which was presented in [17], is strongly related to the notion of *asymptotic* CYK tensor. According to this definition, a spacetime is asymptotically flat if it admits maximal (i.e. 14-dimensional) space of asymptotic solutions of CYK equations. The *asymptotic conformal Yano–Killing tensor* introduced in [17] was analyzed for the Schwarzschild metric in [18], and it was shown that this metric (and other metrics which are asymptotically "Schwarzschildean" up to $O(1/r^2)$ at spatial infinity) is among the metrics fulfilling *strong* asymptotic conditions. It is also clear from the result that 14 asymptotic gravitational charges are well defined on the "Schwarzschildean" background. On the other hand, the concept of the asymptotic CYK tensor which defines conserved quantity at null infinity is only possible for stationary spacetimes. Moreover, the *news function* is an obstruction for the existence of a conserved quantity associated with asymptotic CYK tensor. However, we show in Section 3.2 that one can use CYK tensors from Minkowski background metric and define Bondi four-momentum together with its flux through $\mathscr{I}^+$ in terms of the Weyl tensor. It is also shown that the same construction for angular momentum is possible if we assume that the "ofam" charges are vanishing (see table in Section 3.2).

We allowed us to remind the reader previous results (in the new extended and improved form) about CYK tensors to be able to compare them with our new results which we present in this paper. Here we show the relations between solutions of the spin-2 field equations and Maxwell fields, and apply these relations for the construction of new conserved quantities which are quadratic in terms of the Weyl tensor. The obtained formulae are similar to the functionals obtained with the help of the Bel–Robinson tensor and they should be useful in Christodoulou–Klainerman [6] method to control asymptotic behaviour of the initial data in General Relativity.

It also occurs that our new (quadratic in terms of the Weyl tensor) functionals can be nicely described by a universal object — a new tensor with six indices. We also examine its formal properties. Our proposition is not included in the several generalizations of the Bel–Robinson tensor which are called superenergy tensors (see e.g. [23]). One can say that we propose a new super-tensor with interesting properties.

This paper is organized as follows: In the next Section we remind some basic facts about spin-2 field. Section 3 contains small review about CYK tensors and their applications to the asymptotic charges. In Section 4 we show the relation between spin-2 field and Maxwell field. Section 5 is devoted to the new conserved quantities which are quadratic in terms of the spin-2 field. To clarify the exposition some of the technical results and proofs have been shifted to the appendix. Moreover, in Appendix A we have added, for completeness, some general properties of CYK tensors. We include also the list of symbols in Appendix C.

2 Spin-2 field

Let us start with the standard formulation of a spin-2 field $W_{\mu\alpha\nu\beta}$ in Minkowski spacetime equipped with a flat metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. The field W can be also interpreted as a Weyl tensor for linearized gravity (see [5], [17], [20]).

Definition 1. *The following properties:*

$$W_{\mu\alpha\nu\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]} \ , \ \ W_{\mu[\alpha\nu\beta]} = 0 \ , \ \ \eta^{\mu\nu} W_{\mu\alpha\nu\beta} = 0 \tag{2.1}$$

can be used as a definition of spin-2 field W.

The $*$–operation defined as

$$({}^*W)_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}\,, \quad (W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$$

has the following properties:

$$({}^*W^*)_{\alpha\beta\gamma\delta} = \frac{1}{4}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\gamma\delta}\,, \quad {}^*W = W^*\,, \quad {}^*({}^*W) = {}^*W^* = -W\,,$$

where $\varepsilon_{\mu\nu\gamma\delta}$ is a Levi–Civita skew-symmetric tensor and ${}^*W$ is called dual spin-2 field. The above formulae are also valid for general Lorentzian metrics.

Moreover, Bianchi identities play a role of field equations and we have the following

Lemma 1. *Field equations*

$$\nabla_{[\lambda}W_{\mu\nu]\alpha\beta} = 0 \tag{2.2}$$

are equivalent to

$$\nabla^\mu W_{\mu\nu\alpha\beta} = 0 \ \ or \ \ \nabla_{[\lambda}{}^*W_{\mu\nu]\alpha\beta} = 0 \ \ or \ \ \nabla^\mu\,{}^*W_{\mu\nu\alpha\beta} = 0\,.$$

The equations in the above Lemma are also valid for any Ricci flat metric and its Weyl tensor.

3 Conformal Yano–Killing tensors

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field. Contracting the Weyl tensor $W^{\mu\nu\kappa\lambda}$ with $Q_{\mu\nu}$ we obtain a natural object which can be integrated over two-surfaces. The result does not depend on the choice of the surface if $Q_{\mu\nu}$ fulfills the following condition introduced by Penrose (see [21] and [15]):

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda}Q_{\kappa]}{}^\delta{}_{;\delta} = 0\,. \tag{3.1}$$

Following [17] one can rewrite equation (3.1) in a generalized form for n-dimensional space-time with metric $g_{\mu\nu}$:

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1}g_{\sigma[\lambda}Q_{\kappa]}{}^\delta{}_{;\delta} = 0 \tag{3.2}$$

or in the equivalent form:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^\mu{}_{;\mu}\right)\,. \tag{3.3}$$

Let us define

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^\mu{}_{;\mu}\right) \tag{3.4}$$

Definition 2. *A skew-symmetric tensor $Q_{\mu\nu}$ is a* conformal Yano–Killing tensor *(or simply CYK tensor) for the metric g iff $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.*

The CYK tensor is a natural generalization of the Yano tensor[1] (see [3], [14] and [27]) with respect to the conformal rescalings. More precisely, for any positive scalar function

---

[1]A Yano tensor (often called Killing-Yano) fulfills stronger equation $Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = 0$ (i.e. (3.3) with vanishing right-hand side), hence every Yano tensor is a CYK tensor but not vice versa. The classification of Killing-Yano tensors is given by Dietz and Rudiger in [11] where they also consider canonical line elements of the metrics admitting KY tensors (cf. [9], [16]). A similar problems for CYK tensors seems to be not yet solved. Several interesting applications of Killing-Yano tensors are proposed in [1], [22], [25], [26]. One can also construct a scalar potentials for the Maxwell and massless Dirac equations by using CYK tensors (see [2]).

$\Omega > 0$ and for a given metric $g_{\mu\nu}$ we obtain:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = \Omega^{-3}\mathcal{Q}_{\lambda\kappa\sigma}(\Omega^3 Q, \Omega^2 g). \tag{3.5}$$

The formula (3.5) and the above definition of CYK tensor gives the following

Theorem 1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$ than $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

It is interesting to notice, that a tensor $A_{\mu\nu}$ — a "square" of the CYK tensor $Q_{\mu\nu}$ defined as follows:

$$A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$$

fulfills the following equation:

$$A_{(\mu\nu;\kappa)} = g_{(\mu\nu} A_{\kappa)} \quad \text{with} \quad A_\kappa = \frac{2}{n-1} Q_\kappa{}^\lambda Q_\lambda{}^\delta{}_{;\delta} \tag{3.6}$$

which simply means that the symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor. This can be also described by the following

Theorem 2. *If $Q_{\mu\nu}$ is a skew-symmetric conformal Yano–Killing tensor than $A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$ is a symmetric conformal Killing tensor.*

<u>Remark</u> We show at the end of Appendix A that CYK tensor is a solution of the following conformally invariant equation:

$$(\Box + \frac{1}{6} R) Q = \frac{1}{2} W(Q, \cdot).$$

Moreover, we show in the same Appendix that if Q is a CYK tensor and the metric is Ricci flat then $K^\mu := Q^{\mu\lambda}{}_{;\lambda}$ is a Killing vector field.

For our purposes we need to specify the formulae (3.2) and (3.3) to the special case of the flat four-dimensional Minkowski space ($g_{\mu\nu} = \eta_{\mu\nu}$, $n = 4$). In this simple situation the general CYK tensor assumes the following form in Cartesian coordinates (x^μ):

$$Q^{\mu\nu} = q^{\mu\nu} + 2u^{[\mu}x^{\nu]} - \varepsilon^{\mu\nu}{}_{\kappa\lambda}v^\kappa x^\lambda - \frac{1}{2}k^{\mu\nu}x_\lambda x^\lambda + 2k^{\lambda[\nu}x^{\mu]}x_\lambda, \tag{3.7}$$

where $q^{\mu\nu}$, $k^{\mu\nu}$ are constant skew-symmetric tensors and u^μ, v^μ are constant vectors.

It is easy to verify that the charge given by $Q_{\mu\nu}$ is well defined. Indeed, we have:

$$\int_{\partial V} W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa} \mathrm{d}S_{\mu\nu} = \int_V (W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa})_{,\nu} \, \mathrm{d}\Sigma_\mu =$$

$$= \int_V (W^{\mu\nu\lambda\kappa}{}_{,\nu} Q_{\lambda\kappa} + W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa,\nu}) \mathrm{d}\Sigma_\mu = 0,$$

where the first term vanishes because of the field equations and the second term vanishes because of the symmetries of the Weyl tensor and because of equation (3.1). The above equality implies that the flux of the quantity $W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa}$ through any two closed two-surfaces S_1 and S_2 is the same if there is a three-volume V between them (i.e. if $\partial V = S_1 \cup S_2$). We define the charge corresponding to the specific CYK tensor Q as the value of this flux.

For any skew-symmetric tensor $t_{\lambda\kappa}$ let us define its dual $t^*_{\mu\nu}$ as follows:

$$t^*_{\mu\nu} := \frac{1}{2} \varepsilon_{\mu\nu}{}^{\lambda\kappa} t_{\lambda\kappa}.$$

4

The above construction applied to the dual spin-2 field $^*W$

$$\int_S {}^*W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa} \mathrm{d}S_{\mu\nu} = \int_S W^{\mu\nu\lambda\kappa} Q^*{}_{\lambda\kappa} \mathrm{d}S_{\mu\nu}$$

(cf. (4.3)) does not give more charges because the dual tensor Q^* has the same form (3.7) with the following interchange:

$$q \longleftrightarrow q^* \quad k \longleftrightarrow k^* \quad u \longleftrightarrow v\,.$$

Although, for generic metric, CYK equation is not invariant with respect to the $*$–operation the space of solutions in Minkowski spacetime is closed with respect to Hodge dual. We can summarize this property by the following

Lemma 2. *If Q is a CYK tensor in Minkowski spacetime than the dual tensor Q^* possesses also a CYK property.*

Let us also observe that the solutions of equation (3.7) form a twenty-dimensional vector space. This means that

Lemma 3. *A dimension of the space of CYK tensors in Minkowski spacetime is 20.*

<u>Remark</u> The Theorem 1 implies that the above Lemma is also true for any conformally flat metric.

Let $\mathcal{D}$ be a generator of dilatations in Minkowski spacetime. The generators

$$\mathcal{T}_\mu := \frac{\partial}{\partial x^\mu}\,, \quad \mathcal{L}_{\mu\nu} := x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\,, \quad \mathcal{D} := x^\nu \frac{\partial}{\partial x^\nu} \tag{3.8}$$

of pseudo-similarity group (Poincaré group extended by scaling transformation) obey the following commutation relations:

$$[\mathcal{T}_\mu, \mathcal{T}_\nu] = 0\,, \quad [\mathcal{T}_\mu, \mathcal{L}_{\alpha\beta}] = \eta_{\mu\alpha}\mathcal{T}_\beta - \eta_{\mu\beta}\mathcal{T}_\alpha\,, \quad [\mathcal{T}_\mu, \mathcal{D}] = \mathcal{T}_\mu\,, \quad [\mathcal{D}, \mathcal{L}_{\alpha\beta}] = 0\,,$$

$$[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\alpha\beta}] = \eta_{\mu\alpha}\mathcal{L}_{\beta\nu} - \eta_{\mu\beta}\mathcal{L}_{\alpha\nu} + \eta_{\nu\alpha}\mathcal{L}_{\mu\beta} - \eta_{\nu\beta}\mathcal{L}_{\mu\alpha}\,.$$

The above algebra allows us to define a natural basis in the twenty-dimensional space of CYK tensors in Minkowski spacetime: $\mathcal{T}_\mu \wedge \mathcal{T}_\nu$, $\mathcal{D} \wedge \mathcal{T}_\mu$, $(\mathcal{D} \wedge \mathcal{T}_\mu)^*$, $\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D}, \mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu$ with $\mu < \nu$.

The following conserved quantities has been introduced in [17]:

$$w_{\mu\nu} := \frac{1}{16\pi} \int_{\partial\Sigma} W(\mathcal{T}_\mu \wedge \mathcal{T}_\nu) \tag{3.9}$$

$$w^*_{\mu\nu} := \frac{1}{16\pi} \int_{\partial\Sigma} {}^*W(\mathcal{T}_\mu \wedge \mathcal{T}_\nu) = \frac{1}{16\pi} \int_{\partial\Sigma} W^*(\mathcal{T}_\mu \wedge \mathcal{T}_\nu)$$

$$p_\mu := \frac{1}{16\pi} \int_{\partial\Sigma} W(\mathcal{D} \wedge \mathcal{T}_\mu) \tag{3.10}$$

$$b_\mu := \frac{1}{16\pi} \int_{\partial\Sigma} {}^*W(\mathcal{D} \wedge \mathcal{T}_\mu) \tag{3.11}$$

$$j_{\mu\nu} := \frac{1}{16\pi} \int_{\partial\Sigma} W\left(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D}, \mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu\right) \tag{3.12}$$

$$j^*_{\mu\nu} := \frac{1}{16\pi} \int_{\partial\Sigma} {}^*W \left(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu \right)$$

The conservation law for the charge $w_{\mu\nu}$ is a consequence of field equations:

$$\int_{\partial\Sigma} W^{\mu\nu}{}_{\lambda\kappa}\mathrm{d}\sigma_{\mu\nu} = \int_\Sigma (W^{\mu\nu}{}_{\lambda\kappa}),_\nu \,\mathrm{d}\Sigma_\mu = 0$$

For p^μ and b^μ we obtain the conservation laws from the following observation:

$$\int_{\partial\Sigma} x^\lambda W^{\mu\nu}{}_{\lambda\kappa}\mathrm{d}\sigma_{\mu\nu} = \int_\Sigma \left(x^\lambda W^{\mu\nu}{}_{\lambda\kappa},_\nu + \delta^\lambda_\nu W^{\mu\nu}{}_{\lambda\kappa} \right)\mathrm{d}\Sigma_\mu = 0$$

(the same holds for ${}^*W$).
For $j^{\mu\nu}$ the corresponding identities are as follows:

$$\int_{\partial\Sigma} (x_\lambda W^{\mu\nu\lambda\kappa}x^\delta - x_\lambda W^{\mu\nu\lambda\delta}x^\kappa - \frac{1}{2}x^\lambda x_\lambda W^{\mu\nu\delta\kappa})\mathrm{d}\sigma_{\mu\nu} =$$

$$= \int_\Sigma (x_\lambda W^{\mu\nu\lambda\kappa}x^\delta - x_\lambda W^{\mu\nu\lambda\delta}x^\kappa - \frac{1}{2}x^\lambda x_\lambda W^{\mu\nu\delta\kappa}),_\nu \,\mathrm{d}\Sigma_\mu =$$

$$= \int_\Sigma (x_\lambda W^{\mu\delta\lambda\kappa} - x_\lambda W^{\mu\kappa\lambda\delta} - x_\lambda W^{\mu\lambda\delta\kappa})\mathrm{d}\Sigma_\mu =$$

$$= \int_\Sigma x_\lambda (W^{\mu\delta\lambda\kappa} + W^{\mu\kappa\delta\lambda} + W^{\mu\lambda\kappa\delta})\mathrm{d}\Sigma_\mu = 0 \,.$$

The charge p^μ is called energy-momentum four-vector or shortly four-momentum, $j^{\mu\nu}$ – angular momentum tensor and b^μ – dual four-momentum. Moreover, $w^{\mu\nu}$ vanishes if we assume that $W = O\left(\frac{1}{r^3}\right)$. Hence only 14 charges remain when we pass to the limit at spatial infinity. However, *we don't know any local argument (i.e. using only field equations) for the vanishing of the charge $w^{\mu\nu}$*. We will also show in the sequel that it plays a crucial role for the existence of angular momentum at future null infinity.

3.1 Asymptotic Conformal Yano–Killing tensors at spatial infinity

Let us restrict ourselves to the case of a Lorentzian manifold M of dimension 4. Consider an asymptotically flat spacetime at spatial infinity, fulfilling the Einstein equations. Moreover, suppose that the energy-momentum tensor of the matter vanishes around spatial infinity ("compactly supported sources"). Let us analyze this situation in terms of an asymptotically flat coordinate system. We suppose that there exists an (asymptotically Minkowskian) coordinate system (x^μ):

$$g_{\mu\nu} - \eta_{\mu\nu} = O(r^{-1}) \quad g_{\mu\nu,\lambda} = O(r^{-2})$$

where $r := \left(\sum_{k=1}^3 (x^k)^2\right)^{1/2}$.

For a general asymptotically flat metric we cannot expect that the CYK equation (3.3) admit any solution. Instead, we assume that the tensor $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g)$ (cf. (3.4)) has a certain asymptotic behaviour at spatial infinity

$$\mathcal{Q}_{\mu\nu\lambda} = O(r^{-c}), \quad c > 0 \,. \tag{3.13}$$

Following [18], suppose that the Riemann tensor $R_{\mu\nu\kappa\lambda}$ behaves asymptotically as follows:

$$R_{\mu\nu\kappa\lambda} = O(r^{-2-d}), \quad d > 0 \,.$$

It can be easily checked (see e.g. [15]) that the vacuum Einstein equations imply the equality:

$$\nabla_\lambda \left({}^*R^{*\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \right) = \frac{1}{3} R^{\mu\lambda\alpha\beta} \mathcal{Q}_{\alpha\beta\lambda} \, . \tag{3.14}$$

The left-hand side of (3.14) defines an asymptotic global charge provided that the right-hand side vanishes sufficiently fast at infinity. It is easy to check that, for this purpose, the exponents c, d have to fulfill the inequality $c + d > 1$. In a typical situation when $d = 1$, the above inequality simply means that $c > 0$. In this case a weaker condition is also possible (for example $\mathcal{Q}_{\mu\nu\lambda} = O((\ln r)^{-1-\epsilon})$ with $\epsilon > 0$). Moreover, when Q is a CYK tensor (i.e. $\mathcal{Q}_{\mu\nu\lambda}$ vanishes) the formula (3.14) gives an exact charge (not only asymptotic):

Lemma 4. *If Q is a CYK tensor for the Ricci flat metric g and R is its Riemann tensor than the integral*

$$I(Q) := \int_S {}^*R^{*\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \mathrm{d}S_{\mu\lambda}$$

does not depend on the choice of the closed surface S.

More precisely, it gives the same number for two "spherical" surfaces which can be connected by three-dimensional volume which is located in the Ricci flat region.

Definition 3. *An* Asymptotic Conformal Yano–Killing tensor *(ACYK tensor) for the asymptotically flat metric g is a skew-symmetric tensor $Q_{\mu\nu}$ such that $\mathcal{Q}_{\mu\nu\lambda}(Q, g) \to 0$ at spatial infinity.*

Suppose that $Q_{\mu\nu}$ behaves at spatial infinity as follows $Q_{\mu\nu} = O(r^a)$, $Q_{\lambda\kappa,\sigma} = O(r^{a-1})$.

For constructing the ACYK tensor, we can begin with the solutions of (3.1) in flat Minkowski space. The asymptotic behaviour of ACYK tensors for the flat metric explains the following behaviour in general case:

$$Q_{\mu\nu} = {}^{(2)}Q_{\mu\nu} + {}^{(1)}Q_{\mu\nu} + {}^{(0)}Q_{\mu\nu}$$

where ${}^{(2)}Q_{\mu\nu} = O(r^2)$, ${}^{(1)}Q_{\mu\nu} = O(r)$ and ${}^{(0)}Q_{\mu\nu} = O(r^{1-c})$.

It is easy to verify that $a \leq d$ is sufficient for the convergence of the integral $I(Q)$ from Lemma 4. In particular, for $a = 1 - c$ the integral $I({}^{(0)}Q)$ vanishes if we assume that $d = 1$. Moreover, for $d = a = 1$ any ${}^{(1)}Q_{\mu\nu}$ gives finite limit at spatial infinity for the surface integral $I(Q)$ but the ACYK property is needed to be sure that this limit does not depend on the sequence of two-dim. spheres approaching spatial infinity. These considerations show that the energy-momentum four-vector and the dual one are well defined for any asymptotically flat spacetime.

The situation becomes less trivial when we pass to the case of ${}^{(2)}Q_{\mu\nu}$ which corresponds to angular momentum. The integral $I({}^{(2)}Q)$ for a generic ${}^{(2)}Q_{\mu\nu}$ is divergent but if ACYK property for ${}^{(2)}Q_{\mu\nu}$ is fulfilled it has to converge (the divergent part of the integrand in Lemma 4 integrates to zero). However, the existence of nontrivial ${}^{(2)}Q_{\mu\nu}$ with ACYK property for any asymptotically flat spacetime is not obvious[2] and this is the origin of the difficulties with the definition of the angular momentum.

We proposed in [17] a stronger definition of the asymptotic flatness. This definition is motivated by the above discussion.

---

[2]Let us notice that the first derivatives of ${}^{(1)}Q_{\mu\nu}$ are finite (${}^{(1)}Q_{\mu\nu,\sigma} = O(1)$) so it is easy to believe that for particular ${}^{(1)}Q_{\mu\nu}$ some combinations (in our case $\mathcal{Q}_{\mu\nu\lambda}$) of the first derivatives may have better asymptotics and they vanish at spatial infinity. On the other hand ${}^{(2)}Q_{\mu\nu,\sigma} = O(r)$ and now we need to lower more than one order which is harder.

Suppose that there exists a coordinate system (x^μ) such that:

$$g_{\mu\nu} - \eta_{\mu\nu} = O(r^{-1})\,, \quad \Gamma^\kappa{}_{\mu\nu} = O(r^{-2})\,, \quad R_{\mu\nu\kappa\lambda} = O(r^{-3})\,.$$

In the space of ACYK tensors fulfilling the asymptotic condition

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{3}\left(g_{\sigma\lambda}Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^\mu{}_{;\mu}\right) = \mathcal{Q}_{\lambda\kappa\sigma} = O(r^{-1}) \tag{3.15}$$

we define the following equivalence relation:

$$Q_{\mu\nu} \equiv Q'_{\mu\nu} \iff Q_{\mu\nu} - Q'_{\mu\nu} = O(1) \tag{3.16}$$

for $r \to \infty$. We assume that the space of equivalence classes defined by (3.15) and (3.16) has a finite dimension D as a vector space.

Definition 4. *A spacetime is* strongly *asymptotically flat at spatial infinity iff the number D of gauge equivalence classes of relation (3.16) equals 14 and the total dual four-momentum b^μ vanishes.*

It was shown in [18] that the Schwarzschildean metrics[3] are examples of *strongly* asymptotically flat spacetimes and they possess the full set of 14 asymptotic CYK tensors. The maximal dimension $D = 14$ corresponds to the situation where there are no supertranslation problems in the definition of a total angular momentum at spatial infinity. In the case of spacetimes for which $D < 14$ the lack of certain ACYK tensor means that the corresponding asymptotic charge is not well defined. The example of Kerr–NUT spacetime analyzed in [18] suggests that we should also assume that the asymptotic dual four-momentum b^μ is vanishing.

3.2 Non-conserved charges at future null infinity

The equality (3.14) rewritten in terms of the Weyl tensor for the Einstein vacuum metric can be also used in the radiating regime. Let us fix the conventions related to conformal rescaling (see e.g. [12]) at $\mathscr{I}^+$. The non-physical metric we denote by $\tilde{g}$ and the physical metric by g. They are conformally related as follows

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \quad \Omega^2 \tilde{g}^{\mu\nu} = g^{\mu\nu}\,, \quad \tilde{W}^\lambda{}_{\mu\nu\kappa}(\tilde{g}) = W^\lambda{}_{\mu\nu\kappa}(g)\,,$$

where $\tilde{W}$ is the Weyl tensor for the metric $\tilde{g}$. Moreover, from (3.5) we have

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = \Omega^{-3}\tilde{\mathcal{Q}}_{\lambda\kappa\sigma}(\tilde{Q},\tilde{g})\,, \quad \mathcal{Q}^{\lambda\kappa\sigma}(Q,g) = \Omega^3 \tilde{\mathcal{Q}}^{\lambda\kappa\sigma}(\tilde{Q},\tilde{g})\,,$$

where the corresponding rescaled CYK tensors are defined as follows (cf. Theorem 1)

$$\tilde{Q}_{\alpha\beta} := \Omega^3 Q_{\alpha\beta}\,, \quad \tilde{Q}^{\alpha\beta} := \Omega^{-1}Q^{\alpha\beta}\,.$$

From above formulae we can examine the conformal rescalings of the equation

$$\nabla_\lambda\left(W^{\lambda\mu}{}_{\alpha\beta}Q^{\alpha\beta}\right) = \frac{1}{3}W^{\mu\lambda\alpha\beta}\mathcal{Q}_{\alpha\beta\lambda} \tag{3.17}$$

which is equivalent to (3.14) for Einstein vacuum metric. Let us observe that the right-hand side

$$W^\mu{}_{\lambda\alpha\beta}\mathcal{Q}^{\alpha\beta\lambda}(Q,g) = \Omega^3 \tilde{W}^\mu{}_{\lambda\alpha\beta}\tilde{\mathcal{Q}}^{\alpha\beta\lambda} \tag{3.18}$$

---

[3]e.g. Kerr metric is included in this group. Moreover, according to [7] and [10] there exist a non-trivial class of metrics which are Schwarzschildean.

of the formula (3.17) can be interpreted as the flux of the quantity

$$I(S) := \int_S W^{\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \mathrm{d}S_{\mu\lambda}$$

which is defined by the surface integral (the left-hand side of (3.17)). We assume that two-surface S is a sphere and it is close to null infinity. We have the following transformation rules for the corresponding densities with respect to the conformal rescalings:

$$\sqrt{-\det g}\, W^{\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta}(g) = \sqrt{-\det \tilde{g}}\, \Omega^{-1} \tilde{W}^{\mu\lambda}{}_{\alpha\beta} \tilde{Q}^{\alpha\beta}(\tilde{g})\,. \tag{3.19}$$

Moreover, the flux transforms in the similar way

$$\sqrt{-\det g}\, W^{\mu}{}_{\lambda\alpha\beta} \mathcal{Q}^{\alpha\beta\lambda}(Q,g) = \sqrt{-\det \tilde{g}}\, \Omega^{-1} \tilde{W}^{\mu}{}_{\lambda\alpha\beta} \tilde{\mathcal{Q}}^{\alpha\beta\lambda}\,, \tag{3.20}$$

and the asymptotic behaviour

$$\tilde{W}^{\mu}{}_{\lambda\alpha\beta} = O(\Omega) \quad \tilde{\mathcal{Q}}^{\alpha\beta\lambda} = O(1) = \tilde{Q}^{\alpha\beta} \tag{3.21}$$

gives the finite integrals at $\mathscr{I}^+$. The formulae (3.19-3.20) allow us to define the flux of the energy (or other quantity associated with Q) through the piece of $\mathscr{I}^+$ between any two cross-sections of the null infinity. More precisely, let $s_i : S^2 \longrightarrow \mathscr{I}^+$ for $i = 1, 2$ be two different cross-sections of $\mathscr{I}^+$ such that there exists $N \subset \mathscr{I}^+$ with $\partial N = s_2(S^2) \cup s_1(S^2)$. Then we have

$$I(s_2) - I(s_1) = \int_{\partial N} \sqrt{-\det \tilde{g}}\, \Omega^{-1} \tilde{W}^{\mu\lambda}{}_{\alpha\beta} \tilde{Q}^{\alpha\beta} \partial_\mu \wedge \partial_\lambda \rfloor \mathrm{d}y^0 \wedge \ldots \wedge \mathrm{d}y^3 =$$

$$= \frac{1}{3} \int_N \sqrt{-\det \tilde{g}}\, \Omega^{-1} \tilde{W}^{\mu}{}_{\lambda\alpha\beta} \tilde{\mathcal{Q}}^{\alpha\beta\lambda} \partial_\mu \rfloor \mathrm{d}y^0 \wedge \ldots \wedge \mathrm{d}y^3 = (\text{flux through } N) \tag{3.22}$$

In Minkowski spacetime for $\Omega = x = \frac{1}{r}$ we have

$$\tilde{\eta}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -x^2 \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}x + \breve{h}_{AB} \mathrm{d}y^A \mathrm{d}y^B \tag{3.23}$$

$$\tilde{\eta}^{\mu\nu} \partial_\mu \partial_\nu = x^2 \partial_x \partial_x + 2\partial_u \partial_x + \breve{h}^{AB} \partial_A \partial_B \tag{3.24}$$

and the corresponding CYK tensors take the following form:

charge	Poincaré	Q	$\tilde{Q}$
ofam	—	$\mathcal{T}_k \wedge \mathcal{T}_0$	$v_{,A} \mathrm{d}y^A \wedge \mathrm{d}x$
ofam	—	$\mathcal{T}_k \wedge \mathcal{T}_l$	$\varepsilon_A{}^B \hat{v}_{,B} \mathrm{d}y^A \wedge \mathrm{d}x$
energy	time transl.	$\mathcal{D} \wedge \mathcal{T}_0$	$\mathrm{d}u \wedge \mathrm{d}x$
linear mom.	space transl.	$\mathcal{D} \wedge \mathcal{T}_k$	$v\mathrm{d}u \wedge \mathrm{d}x$
angular momentum	rotation	$\mathcal{D} \wedge \mathcal{L}_{kl} - \frac{1}{2} x^\mu x_\mu \mathcal{T}_k \wedge \mathcal{T}_l$	$\varepsilon_A{}^B \hat{v}_{,B} \mathrm{d}y^A \wedge (x^{-1}\mathrm{d}x + \mathrm{d}u)$ $+\frac{1}{2} \hat{v} \varepsilon_{AB} \mathrm{d}y^A \wedge \mathrm{d}y^B$
static moment	boost	$\mathcal{D} \wedge \mathcal{L}_{0k} - \frac{1}{2} x^\mu x_\mu \mathcal{T}_0 \wedge \mathcal{T}_k$	$v_{,A} \mathrm{d}y^A \wedge (x^{-1}\mathrm{d}x + \mathrm{d}u)$

where $v = \frac{x_k}{r}$, $\hat{v} = \varepsilon_{klm} \frac{x^m}{r}$ are dipole functions on a sphere S^2 which is parameterized by co-ordinates y^A. The above asymptotics (at future null infinity) for CYK tensors in Minkowski spacetime, provided for any asymptotically flat manifold, suggest that the energy-momentum

four-vector and its density of flux (3.20) is always finite ($\tilde{Q} = O(1)$) but for angular momentum, in general, we may have divergences ($\tilde{Q} = O(\Omega^{-1})$). The extra condition for finiteness of angular momentum at $\mathscr{I}^+$ is related with the term:

$$x^{-1}\tilde{W}^{\mu\lambda\alpha\beta}\tilde{Q}_{\alpha\beta} = \begin{cases} x^{-2}\tilde{W}^{uxAx}v_{,A} + O(1) & \text{for boost} \\ x^{-2}\tilde{W}^{uxAx}\varepsilon_A{}^B\hat{v}_{,B} + O(1) & \text{for rotation} \end{cases}$$

and it becomes finite if we assume that dipole part[4]

$$\text{dip}\left(\tilde{W}^{uxAx}\right) = O(x^2). \tag{3.25}$$

This also means that ofam-charges are **obstructions** **f**or the existence of **a**ngular **m**omentum at $\mathscr{I}^+$. The asymptotic conditions which guarantee finiteness of the four-momentum at future null infinity are as follows: energy — $\text{mon}\left(x^{-1}\tilde{W}^{uxux}\right) = O(1)$[5], linear momentum — $\text{dip}\left(x^{-1}\tilde{W}^{uxux}\right) = O(1)$ and they are fulfilled for any asymptotically flat spacetime satisfying (3.21). However, for angular momentum the asymptotic condition $\text{dip}\left(x^{-2}\tilde{W}^{uxAx}\right) = O(1)$ is an extra assumption which is fulfilled for Bondi metrics (see [8] p. 91, [24]) but it is not obvious (see also [15]). More precisely, for Bondi metric in the asymptotic form given in Section 5.6 of [8] we obtain

$$x^{-1}\tilde{W}^{uxAx} = x^{-1}\tilde{g}^{xu}\tilde{g}^{xu}\tilde{g}^{AB}\tilde{W}^u{}_{uBu} + \text{higher order terms}$$

and the main asymptotic term takes the following form

$$x^{-1}\tilde{W}^{uxAx} \sim x^{-1}h^{AB}W^u{}_{uBu} \sim -x^{-2}\partial_u U^A = \frac{1}{2}\partial_u\chi^{AB}{}_{||B} + O(x).$$

Hence, the asymptotics of $x^{-1}\tilde{W}^{uxAx}$ at $\mathscr{I}^+$ corresponds to $\frac{1}{2}\partial_u\chi^{AB}{}_{||B}$ and the dipole part of $\chi^{AB}{}_{||B}$ vanishes because χ_{AB} is a traceless symmetric tensor on a unit sphere (see also Appendix E in [20]).

The corresponding components of the Weyl field in linearized gravity derived from (B.1) (presented in Appendix) are the following:

$$2\tilde{W}^{uxxu} = \mathbf{x}, \quad 2^*\tilde{W}^{uxxu} = \mathbf{y},$$

$$2\tilde{W}^{xuxA}{}_{||A} = \partial_u\mathbf{x}, \quad 2\tilde{W}^{xux}{}_{A||B}\varepsilon^{AB} = \partial_u\mathbf{y}$$

Moreover, in linearized theory

$$\text{dip}\left(\partial_u\mathbf{x}\right) = 0 = \text{dip}\left(\partial_u\mathbf{y}\right).$$

The condition (3.25) means that zero from the linear case should be replaced (in nonlinear case) by higher order asymptotics.

4 Spin-2 field + CYK tensor $\longrightarrow$ Maxwell field

Let us define a skew-symmetric tensor

$$F_{\mu\nu}(W, Q) := W_{\mu\nu\lambda\kappa}Q^{\lambda\kappa}, \tag{4.1}$$

where W is the spin-2 field and Q is the CYK tensor.

---

[4]The dipole part of a vector field on S^2 is its orthogonal projection onto the six-dimensional space of conformal vector fields which is simultaneously a "first" eigenspace (with unit eigenvalue) for Laplace–Beltrami operator $\mathbf{\Delta}$ (see also Appendix E in [20]).

[5]By mon we denote monopole part of a scalar field on S^2 i.e. its orthogonal projection onto the one-dimensional kernel of Laplace–Beltrami operator $\mathbf{\Delta}$ (see also Appendix E in [20]).

Theorem 3. *For any spin-2 field W satisfying field equations (2.2) and any CYK tensor Q in Minkowski spacetime the skew-symmetric tensor $F_{\mu\nu}$ (two-form F) defined by (4.1) fulfills vacuum Maxwell equations i.e.*

$$\mathrm{d}F = 0 = \mathrm{d}F^* \quad \Longleftrightarrow \quad \nabla_\lambda F^{*\mu\lambda} = 0 = \nabla_\lambda F^{\mu\lambda},$$

where $F^{*\mu\lambda} = \frac{1}{2}\varepsilon^{\mu\lambda\rho\sigma}F_{\rho\sigma}$.

Proof. This is a simple consequence of the spin-2 field equations and the definition of CYK tensor. More precisely, from (3.14) we have

$$0 = -\nabla_\lambda \left({}^*W^{*\mu\lambda}{}_{\alpha\beta}Q^{\alpha\beta} \right) = \nabla_\lambda \left(W^{\mu\lambda}{}_{\alpha\beta}Q^{\alpha\beta} \right) = \nabla_\lambda F^{\mu\lambda}(W, Q), \tag{4.2}$$

so half of Maxwell equations are proved. Moreover,

$$F^{*\mu\lambda}(W, Q) = {}^*W^{\mu\lambda}{}_{\alpha\beta}Q^{\alpha\beta} = W^{*\mu\lambda}{}_{\alpha\beta}Q^{\alpha\beta} = W^{\mu\lambda}{}_{\alpha\beta}Q^{*\alpha\beta} = F^{\mu\lambda}(W, Q^*) \tag{4.3}$$

and if Q is a CYK tensor than Q^* is also a CYK tensor (Lemma 2) hence from (4.2) and (4.3) we get the second half of Maxwell equations:

$$0 = \nabla_\lambda F^{\mu\lambda}(W, Q^*) = \nabla_\lambda F^{*\mu\lambda}(W, Q). \qquad \square$$

This way for each spin-2 field we can assign 20 linearly independent Maxwell fields. Each of them may carry electric charge which is described by (3.9–3.12). Moreover, we can define standard energy-momentum tensor for each of them which is obviously quadratic in terms of F so it would be also quadratic in terms of W. This phenomenon is presented in the next section.

The Theorem 3 can be generalized for Ricci flat metric, but we have to add an assumption that Q^* is a CYK tensor because the Lemma 2 is no longer valid in this case.

5 Conserved quantities as quadratic polynomials in terms of spin-2 field

Let us start with the standard definition of energy-momentum tensor for Maxwell field F:

$$T^{\mathrm{EM}}_{\mu\nu}(F) := \frac{1}{2}\left(F_{\mu\sigma}F_\nu{}^\sigma + F^*{}_{\mu\sigma}F^*{}_\nu{}^\sigma \right) = F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} \tag{5.1}$$

The energy-momentum tensor $T^{\mathrm{EM}}_{\mu\nu}(F)$ is symmetric, traceless and satisfies the following positivity condition:
for any non-spacelike future-directed vector fields X, Y we have $T^{\mathrm{EM}}_{\mu\nu}(F)X^\mu Y^\nu \geq 0$.
Straightforward from the definition we get

$$T^{\mathrm{EM}}_{\mu\nu}(F) = T^{\mathrm{EM}}_{\mu\nu}(F^*). \tag{5.2}$$

Moreover, if F is a Maxwell field than

$$\nabla^\mu T^{\mathrm{EM}}_{\mu\nu}(F) = 0, \tag{5.3}$$

and if X is a conformal Killing vector field than the quantity

$$CQ^{EM}(X, \Sigma; F) := \int_\Sigma T^{\mathrm{EM}}_{\mu\nu}X^\mu \mathrm{d}\Sigma^\nu$$

defines a global conserved quantity for the spacelike hypersurface Σ with the end at spacelike infinity.

11

Let us restrict ourselves to the spacelike hyperplanes $\Sigma_t := \{x \in M : x^0 = t = \text{const}\}$. We use the following convention for indices: (x^μ) $\mu = 0, \ldots, 3$ are Cartesian coordinates in Minkowski spacetime, x^0 denotes temporal coordinate and (x^k) $k = 1, 2, 3$ are coordinates on the spacelike surface Σ_t. If the quantity $CQ^{EM}(X, \Sigma_t)$ is finite for $t = 0$ than it is constant in time. If we want to get a positive definite integral CQ^{EM}, we have to restrict ourselves to the case of non-spacelike field X. We can choose time translation $\mathcal{T}_0$ or time-like conformal acceleration $\mathcal{K}_0$, where

$$\mathcal{K}_\mu := -2x_\mu \mathcal{D} + x^\sigma x_\sigma \mathcal{T}_\mu \tag{5.4}$$

is a set of four "pure" conformal Killing vector fields which should be added to the eleven fields 3.8 to obtain the full 15-dimensional algebra of conformal group. This way we get

Theorem 4. *There are two conserved (in time) positive definite integrals $CQ^{EM}(\mathcal{T}_0, \Sigma_t; F)$ and $CQ^{EM}(\mathcal{K}_0, \Sigma_t; F)$ for the field F satisfying vacuum Maxwell equations.*

Proof. This is a simple consequence of (5.3) and traceless property of T^{EM} which implies $\nabla^\mu \left(T^{EM}_{\mu\nu} X^\nu \right) = 0$ for any conformal Killing vector field X. $\square$

Following [5], for the Bel–Robinson tensor defined as follows

$$T^{BR}_{\mu\nu\kappa\lambda} := W_{\mu\rho\kappa\sigma} W_\nu{}^\rho{}_\lambda{}^\sigma + W^*{}_{\mu\rho\kappa\sigma} W^*{}_\nu{}^\rho{}_\lambda{}^\sigma \tag{5.5}$$

$$= W_{\mu\rho\kappa\sigma} W_\nu{}^\rho{}_\lambda{}^\sigma + W_{\mu\rho\lambda\sigma} W_\nu{}^\rho{}_\kappa{}^\sigma - \frac{1}{8} g_{\mu\nu} g_{\kappa\lambda} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta}, \tag{5.6}$$

where W is a spin-2 field, one can find a natural generalization of the Theorem 4 which is a consequence of the properties similar to 5.3. More precisely, $T^{BR}_{\mu\nu\kappa\lambda}$ is symmetric and traceless in all pairs of indices. Moreover,

$$T^{BR}(W) = T^{BR}(W^*),$$

and if W is a spin-2 field than

$$\nabla^\mu T^{BR}_{\mu\nu\lambda\kappa}(W) = 0. \tag{5.7}$$

If X, Y, Z are conformal Killing vector fields than the quantity

$$CQ^{BR}(X, Y, Z, \Sigma_t; W) := \int_{\Sigma_t} T^{BR}_{\mu\nu\lambda\kappa} X^\mu Y^\nu Z^\lambda \mathrm{d}\Sigma^\kappa$$

defines a global charge at time t. The quantity $T^{BR}(X, Y, Z, T)$ is non-negative for any non-spacelike future-directed vector fields X, Y, Z, T whenever at most two of the vector fields are distinct. From above properties we obtain an extension of the Theorem 4 for the case of spin-2 field W:

Theorem 5. *There are four conserved (in time) positive definite integrals $CQ^{BR}(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0, \Sigma_t; W)$, $CQ^{BR}(\mathcal{K}_0, \mathcal{T}_0, \mathcal{T}_0, \Sigma_t; W)$, $CQ^{BR}(\mathcal{K}_0, \mathcal{K}_0, \mathcal{T}_0, \Sigma_t; W)$ and $CQ^{BR}(\mathcal{K}_0, \mathcal{K}_0, \mathcal{K}_0, \Sigma_t; W)$ for the spin-2 field W satisfying field equations (2.2).*

Proof. Similarly as in the Theorem 4, from (5.7) and traceless property of T^{BR} we get $\nabla^\mu \left(T^{BR}_{\mu\nu\lambda\kappa} X^\kappa Y^\nu Z^\lambda \right) = 0$ for any conformal Killing vector fields X, Y, Z. $\square$

Although the last integral $CQ^{BR}(\mathcal{K}_0, \mathcal{K}_0, \mathcal{K}_0, \Sigma_t; W)$ was not considered by Christodoulou and Klainerman, it seems to be natural to include this quantity in the above Theorem.

We propose the following generalization of the above considerations, namely, let us apply Maxwell field F defined by (4.1) into Theorem 4. Hence for any spin-2 field W satisfying field

12

equations and for each CYK tensor Q we obtain two conserved positive definite quantities: $CQ^{EM}(\mathcal{T}_0, \Sigma_t; F(W, Q))$ and $CQ^{EM}(\mathcal{K}_0, \Sigma_t; F(W, Q))$. Because of the duality property (5.2) for T^{EM} and Lemma 2 the number of the functionals CQ^{EM} reduces to $2 \cdot 20/2 = 20$. We will show in the sequel that not all of them are independent and they fulfill some relations (cf. (5.17-5.20)). Let us denote by Θ the following functional:

$$\Theta(X, V; Q) := \int_{V \subset \Sigma} T^{\text{EM}}_{\mu\nu}\big(F(W, Q)\big) X^\mu \mathrm{d}\Sigma^\nu \tag{5.8}$$

It seems natural to consider the following question:

what is the relation between four conserved quantities CQ^{BR} from Theorem 5 and our functionals Θ?

The answer is very simple:

Theorem 6. *The four conserved quantities CQ^{BR} from Theorem 5 are contained in our functionals Θ.*

This problem can be easier analyzed if we pass to 3+1-decomposition of the spin-2 field. The ten independent components of W split into two 3-dimensional traceless tensors: electric part

$$E(X, Y) := W(X, \mathcal{T}_0, \mathcal{T}_0, Y)$$

and magnetic part

$$H(X, Y) := {}^*W(X, \mathcal{T}_0, \mathcal{T}_0, Y),$$

and the full spin-2 field W expresses in terms of E and H as follows:

$$W_{0kl0} = E_{kl}, \quad W_{0kij} = H_{kl}\varepsilon^l{}_{ij}, \quad W_{klmn} = \varepsilon^i{}_{kl}\varepsilon^j{}_{mn}E_{ij}. \tag{5.9}$$

Let us notice that on the surface $\Sigma_0 = \{x^0 = t = 0\}$ the vector fields $\mathcal{K}_0$ and $\mathcal{T}_0$ are parallel:

$$\mathcal{K}_0(t = 0) = r^2\mathcal{T}_0 = r^2\partial_t$$

and obviously the CYK tensor $\mathcal{D} \wedge \mathcal{T}_0 = r\partial_r \wedge \partial_t$ where r is a radial coordinate. The above observations give the following result for our integrands on the surface $t = 0$:

$$T^{\text{EM}}\big(\mathcal{K}_0, \mathcal{T}_0, F(W, Q)\big) = r^2 T^{\text{EM}}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, Q)\big) \tag{5.10}$$

$$T^{\text{EM}}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{T}_i \wedge \mathcal{T}_0)\big) = \frac{1}{2}\sum_{k=1}^3 \big(E_{ki}E^k{}_i + H_{ki}H^k{}_i\big) \tag{5.11}$$

$$T^{\text{EM}}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{D} \wedge \mathcal{T}_0)\big) = \frac{1}{2}r^2 \big(E_{kr}E^k{}_r + H_{kr}H^k{}_r\big) \tag{5.12}$$

$$T^{\text{EM}}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{D} \wedge \mathcal{T}_i)\big) = \frac{1}{2}r^2 v_{,A}v_{,B}\varepsilon^{AC}\varepsilon^{BD}\big(E_{kC}E^k{}_D + H_{kC}H^k{}_D\big) \tag{5.13}$$

where $v = \frac{x_i}{r}$.

$$T^{\text{EM}}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{D} \wedge \mathcal{L}_{0i} - \frac{1}{2}x^\mu x_\mu \mathcal{T}_0 \wedge \mathcal{T}_i)\big) = \tag{5.14}$$

$$(vE^k{}_r - rv_{||A}E^{kA})(vE_{kr} - rv_{||B}E_k{}^B) + (vH^k{}_r - rv_{||C}H^{kC})(vH_{kr} - rv_{||D}H_k{}^D)$$

13

$$\begin{aligned}
T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0\big) &= r^2 T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0\big) = r^4 T^{BR}\big(\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0\big) \\
&= r^6 T^{BR}\big(\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0\big)
\end{aligned} \tag{5.15}$$

$$T^{BR}\big(\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0\big) = E_{kl}E^{kl} + H_{kl}H^{kl}$$

In Appendix B we present the details how the conserved quantities CQ^{BR} are included in Θ. In particular, the above formulae integrated by parts on each sphere $S(r)$ give the following relations between corresponding functionals:

$$\int_{\Sigma_0} T^{BR}\big(\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0\big)\mathrm{d}^3x = 2\sum_{k=1}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{T}_0,\mathcal{T}_0,F(W,\mathcal{T}_k\wedge\mathcal{T}_0)\big)\mathrm{d}^3x \tag{5.16}$$

$$\int_{\Sigma_0} T^{BR}\big(\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0,\mathcal{T}_0\big)\mathrm{d}^3x = 2\sum_{k=1}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{K}_0,\mathcal{T}_0,F(W,\mathcal{T}_k\wedge\mathcal{T}_0)\big)\mathrm{d}^3x \tag{5.17}$$

$$= 2\sum_{\mu=0}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{T}_0,\mathcal{T}_0,F(W,\mathcal{T}_\mu\wedge\mathcal{D})\big)\mathrm{d}^3x \tag{5.18}$$

$$\int_{\Sigma_0} T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0\big)\mathrm{d}^3x = 2\sum_{\mu=0}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{K}_0,\mathcal{T}_0,F(W,\mathcal{T}_\mu\wedge\mathcal{D})\big)\mathrm{d}^3x \tag{5.19}$$

$$= 2\sum_{i=1}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{T}_0,\mathcal{T}_0,F(W,\mathcal{D}\wedge\mathcal{L}_{0i}-\tfrac{1}{2}x^\mu x_\mu \mathcal{T}_0\wedge\mathcal{T}_i)\big)\mathrm{d}^3x \tag{5.20}$$

$$\int_{\Sigma_0} T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0\big)\mathrm{d}^3x = 2\sum_{i=1}^{3}\int_{\Sigma_0} T^{EM}\big(\mathcal{K}_0,\mathcal{T}_0,F(W,\mathcal{D}\wedge\mathcal{L}_{0i}-\tfrac{1}{2}x^\mu x_\mu \mathcal{T}_0\wedge\mathcal{T}_i)\big)\mathrm{d}^3x\,. \tag{5.21}$$

The formulae (5.16-5.21) imply the Theorem 6. It is also clear from (5.17-5.20) that not all twenty functionals Θ are linearly independent. Although we have only checked the linear dependence at $t=0$ the conservation law implies that they are related in the same way at any time t (provided they are finite).

<u>Remark</u> The following problem should be examined: *How many independent functionals Θ do exist for a generic spin-2 field?*

We leave this problem opened, however, we show below (for some examples of Θ functionals) the method which should lead to the answer to the above question.

We should remember that the electric and magnetic tensors on Σ are not free, they are constrained by the following equations

$$E^{kl}{}_{|l} = 0\,, \quad H^{kl}{}_{|l} = 0 \tag{5.22}$$

which are simply tangent to Σ parts of spin-2 field equations (2.2). This means that one should examine the integrals Θ and CQ^{BR} in terms of the unconstrained degrees of freedom which are no longer constrained. This can be done in a systematic way, using quasi-local variables $\mathbf{x} = 2W\big(\mathcal{T}_0,\mathcal{D},\mathcal{D},\mathcal{T}_0\big)$ and $\mathbf{y} = 2{}^*W\big(\mathcal{T}_0,\mathcal{D},\mathcal{D},\mathcal{T}_0\big)$ which are extensively used in [19] and [20]. The full spin-2 field W on surface Σ expresses in terms of Cauchy data $\mathbf{x},\partial_0\mathbf{x},\mathbf{y},\partial_0\mathbf{y}$ due to the following Theorem proposed in [20]:

Theorem 7. *The linearized Riemann tensor for the vacuum Einstein equations depends quasilocally on the invariants $(\mathbf{x},\mathbf{y})$ which contain the full information about the linearized gravitational field. Moreover, the invariants $\mathbf{x}$ and $\mathbf{y}$ fulfill usual wave equation.*

14

The precise form of the Weyl field is given in Appendix B. In particular, electric part E_{kl} expresses in terms of $\mathbf{x}, \partial_0 \mathbf{y}$:

$$r^2 E_{rr} = -r^2 \eta^{AB} E_{AB} = \frac{1}{2}\mathbf{x}, \quad r^2 \varepsilon^{AB} E_{rB||A} = \frac{1}{2}\partial_0 \mathbf{y}, \quad r^3 E^{rA}{}_{||A} = -\frac{1}{2}\partial_r(r\mathbf{x})$$

and two-dimensional traceless part $\overset{\circ}{E}_{AB} := E_{AB} - \frac{1}{2}\eta_{AB}\eta^{CD}E_{CD}$ is given by (B.7) and (B.8). Similarly magnetic part H_{kl} depends on $\mathbf{y}, \partial_0 \mathbf{x}$:

$$r^2 H_{rr} = -r^2 \eta^{AB} H_{AB} = \frac{1}{2}\mathbf{y}, \quad r^2 \varepsilon^{AB} H_{rB||A} = \frac{1}{2}\partial_0 \mathbf{x}, \quad r^3 H^{rA}{}_{||A} = -\frac{1}{2}\partial_r(r\mathbf{y})$$

and traceless part $\overset{\circ}{H}_{AB}$ is given by (B.12) and (B.13). Hence all these components of the spin-2 field inserted into T^{BR} and integrated by parts (see Appendix B) give the following result:

$$
\begin{aligned}
CQ_0^{BR} \quad &:= \quad \int_V T^{BR}\big(\mathcal{K}_0, \mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\big)\mathrm{d}^3 x \qquad\qquad\qquad\qquad (5.23)\\[4pt]
&= \quad \int_V r^2 \big(E_{kl}E^{kl} + H_{kl}H^{kl}\big)r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \qquad\qquad (5.24)\\[4pt]
&= \quad \frac{1}{2}\int_V \Big((r\dot{\mathbf{x}})(-\boldsymbol{\Delta})^{-1}(r\dot{\mathbf{x}}) + \partial_r(r\mathbf{x})(-\boldsymbol{\Delta})^{-1}\partial_r(r\mathbf{x}) + \mathbf{x}^2 \\
&\qquad +(r\dot{\mathbf{y}})(-\boldsymbol{\Delta})^{-1}(r\dot{\mathbf{y}}) + \partial_r(r\mathbf{y})(-\boldsymbol{\Delta})^{-1}\partial_r(r\mathbf{y}) + \mathbf{y}^2\Big)\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\ +\\
&\qquad +\frac{1}{2}\int_V \Big[\partial_r(r^2\dot{\mathbf{x}})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_r(r^2\dot{\mathbf{x}}) + \frac{1}{2}\mathbf{x}^2 \qquad\qquad (5.25)\\
&\qquad +\big(\partial_r[r\partial_r(r\mathbf{x})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{x}\big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\big(\partial_r[r\partial_r(r\mathbf{x})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{x}\big)\\
&\qquad +\big(\partial_r[r\partial_r(r\mathbf{y})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{y}\big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\big(\partial_r[r\partial_r(r\mathbf{y})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{y}\big)\\
&\qquad +\partial_r(r^2\dot{\mathbf{y}})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_r(r^2\dot{\mathbf{y}}) + \frac{1}{2}\mathbf{y}^2\Big]\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}
$$

where by dot we have denoted as usual time derivative and similarly we analyze:

$$
\begin{aligned}
\Theta_0 \quad &:= \quad \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{D}\wedge\mathcal{T}_0)\big)\mathrm{d}^3 x \\[4pt]
&= \quad \frac{1}{2}\int_V r^2 \big(E_{kr}E^{kr} + H_{kr}H^{kr}\big)r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\\[4pt]
&= \quad \frac{1}{4}\int_V \Big(\partial_0(r\mathbf{x})(-\boldsymbol{\Delta})^{-1}\partial_0(r\mathbf{x}) + \partial_r(r\mathbf{x})(-\boldsymbol{\Delta})^{-1}\partial_r(r\mathbf{x}) + \mathbf{x}^2 \qquad (5.26)\\
&\qquad +\partial_0(r\mathbf{y})(-\boldsymbol{\Delta})^{-1}\partial_0(r\mathbf{y}) + \partial_r(r\mathbf{y})(-\boldsymbol{\Delta})^{-1}\partial_r(r\mathbf{y}) + \mathbf{y}^2\Big)\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,.
\end{aligned}
$$

The above reduced expressions for the close to each other quadratic functionals[6] show the differences between them, in particular, CQ_0^{BR} contains second derivatives of the reduced data $\mathbf{x}, \mathbf{y}$. This is also a typical attribute of all functionals associated to the densities (5.11-5.14) except Θ_0 built from (5.12). In our opinion the functional (5.26) seems to be the most natural one because it contains only first (radial and time) derivative of our quasi-local variables. Moreover, one can easily check the conservation law for compactly supported data straightforward from the wave equation which is fulfilled by our quasi-local unconstrained

---

[6]The other expressions may differ also by certain power of r according to (5.10) and (5.15).

degrees of freedom (see Appendix B). The functional Θ_0 is also very close to the energy functional proposed in [19] which takes the following form in Minkowski spacetime:

$$
\begin{aligned}
8\pi\mathcal{H} \quad = \quad & \frac{1}{4}\int_\Sigma\Big[(r\dot{\mathbf{x}})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\dot{\mathbf{x}}) + (r\dot{\mathbf{y}})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\dot{\mathbf{y}}) + \\
& + (r\mathbf{x})_{,3}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\mathbf{x})_{,3} - \mathbf{x}(\boldsymbol{\Delta}+2)^{-1}\mathbf{x} + \\
& + (r\mathbf{y})_{,3}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\mathbf{y})_{,3} - \mathbf{y}(\boldsymbol{\Delta}+2)^{-1}\mathbf{y}\Big]\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,.
\end{aligned}
\tag{5.27}
$$

The integrals (5.26) and (5.27) differ by the operator $(\boldsymbol{\Delta}+2)^{-1}$, hence for each spherical mode (i.e. after spherical harmonics decomposition) they are proportional to each other.

In nonlinear case the quadratic functionals (5.8) should be useful in the so called exterior initial value problems $(V = \Sigma \setminus B(0,R))$ and they should allow to control asymptotic behaviour of the various components of the Weyl tensor.

5.1 Natural (super-)tensor

Let us consider a tensor

$$
T_{\mu\nu\alpha\beta\gamma\delta} := \frac{1}{2}\left(W_{\mu\sigma\alpha\beta}W_\nu{}^\sigma{}_{\gamma\delta} + W_{\mu\sigma\gamma\delta}W_\nu{}^\sigma{}_{\alpha\beta} + W^*{}_{\mu\sigma\alpha\beta}W^*{}_\nu{}^\sigma{}_{\gamma\delta} + W^*{}_{\mu\sigma\gamma\delta}W^*{}_\nu{}^\sigma{}_{\alpha\beta}\right) \tag{5.28}
$$

which is naturally related to our new conserved quantities by the following equality

$$
T^{EM}_{\mu\nu}(F(Q)) = \frac{1}{2}T_{\mu\nu\alpha\beta\gamma\delta}Q^{\alpha\beta}Q^{\gamma\delta}\,.
$$

Tensor T has the following properties:

$$
T_{\mu\nu\alpha\beta\gamma\delta} = T_{\mu\nu\gamma\delta\alpha\beta} = T_{(\mu\nu)[\alpha\beta][\gamma\delta]}\,, \quad T_{\mu\nu\alpha\beta\gamma\delta}g^{\mu\nu} = 0 \tag{5.29}
$$

which are simple consequences of the definition (5.28) and spin-2 field properties. Moreover, T is related with Bel–Robinson tensor as follows

$$
g^{\beta\delta}T_{\mu\nu\alpha\beta\gamma\delta} = T^{BR}_{\mu\nu\alpha\gamma}\,.
$$

One can also show the following properties of tensor T:

$$
\nabla^\mu T_{\mu\nu\alpha\beta\gamma\delta} = 0\,, \quad T_{\mu[\nu\alpha\beta]\gamma\delta} = 0\,. \tag{5.30}
$$

Proof. The divergence-free property for T is a consequence of spin-2 field equations which simultaneously hold for W and W^*, hence we get

$$
\nabla^\mu T_{\mu\nu\alpha\beta\gamma\delta} = \frac{1}{4}\nabla_\nu\left(W^{\mu\sigma}{}_{\alpha\beta}W_{\mu\sigma\gamma\delta} + W^{*\mu\sigma}{}_{\alpha\beta}W^*{}_{\mu\sigma\gamma\delta}\right) = 0
$$

where the last equality is a consequence of the following formula

$$
W^{\mu\sigma}{}_{\alpha\beta}W_{\mu\sigma\gamma\delta} + W^{*\mu\sigma}{}_{\alpha\beta}W^*{}_{\mu\sigma\gamma\delta} = 0
$$

which is equivalent to traceless attribute of T in (5.29) and can be easily checked from properties of spin-2 field with respect to $*$–operation (e.g. $*^2 = -1$). The second equality in (5.30) is implied by the Bianchi identity for W and W^*. $\qquad\square$

The above properties of tensor T allow us to check the following final

Theorem 8. *If P,Q are CYK tensors, X is a conformal vector field and T obeys the properties (5.29) and (5.30) then*

$$
\nabla^\mu\left(T_{\mu\nu\alpha\beta\gamma\delta}X^\nu P^{\alpha\beta}Q^{\gamma\delta}\right) = 0\,.
$$

16

Acknowledgments

A   General properties of conformal Yano–Killing tensors

In this appendix, following [18], we remind some general identities for Killing vector fields and conformal Killing vectors together with the CYK tensors properties.

Let M be a differential manifold of dimension $n > 1$ with a riemannian or pseudoriemannian metric $g_{\mu\nu}$. By ";" we denote the covariant derivative associated with the Levi–Civita connection, and by $R^{\sigma}{}_{\mu\lambda\nu}$ we denote the curvature Riemann tensor. $R_{\mu\nu}$ is the Ricci tensor.

If we have a tensor object $T_{...\mu\nu...}$ then by $T_{...(\mu\nu)...}$ we denote the symmetric part and by $T_{...[\mu\nu]...}$ the skewsymmetric part of $T_{...\mu\nu...}$ with respect to the pair of indices $\mu\nu$. The indices are raised and lowered with respect to the metric $g_{\mu\nu}$ or its inverse.

A.1   Killing vectors

The Killing vector field X^{μ} on M can be defined as a solution of the following equation:

$$X_{\lambda;\mu} + X_{\mu;\lambda} = 0 \,. \tag{A.1}$$

Let us write explicitly three similar identities for any covector field X_{μ}:

$$X_{\lambda;\mu\nu} - X_{\lambda;\nu\mu} = X_{\sigma} R^{\sigma}{}_{\lambda\mu\nu}$$

$$X_{\mu;\lambda\nu} - X_{\mu;\nu\lambda} = X_{\sigma} R^{\sigma}{}_{\mu\lambda\nu}$$

$$X_{\nu;\lambda\mu} - X_{\nu;\mu\lambda} = X_{\sigma} R^{\sigma}{}_{\nu\lambda\mu}$$

From the above equalities we can derive as follows:

$$-2X_{\mu;\nu\lambda} = (X_{\lambda;\mu\nu} + X_{\mu;\lambda\nu}) - (X_{\lambda;\nu\mu} + X_{\nu;\lambda\mu}) + (X_{\nu;\mu\lambda} + X_{\mu;\nu\lambda}) - 2X_{\mu;\nu\lambda} =$$

$$= (X_{\lambda;\mu\nu} - X_{\lambda;\nu\mu}) + (X_{\mu;\lambda\nu} - X_{\mu;\nu\lambda}) - (X_{\nu;\lambda\mu} - X_{\nu;\mu\lambda}) =$$

$$= X_{\sigma} \left(R^{\sigma}{}_{\lambda\mu\nu} + R^{\sigma}{}_{\mu\lambda\nu} - R^{\sigma}{}_{\nu\lambda\mu} \right) =$$

$$= -X_{\sigma} \left(R^{\sigma}{}_{\lambda\mu\nu} + R^{\sigma}{}_{\mu\nu\lambda} + R^{\sigma}{}_{\nu\lambda\mu} \right) + 2X_{\sigma} R^{\sigma}{}_{\lambda\mu\nu} = -2X_{\sigma} R^{\sigma}{}_{\lambda\nu\mu} \,.$$

The above manipulations lead to the following second order equation:

$$X_{\mu;\nu\lambda} = X_{\sigma} R^{\sigma}{}_{\lambda\nu\mu} \,. \tag{A.2}$$

The trace of the above equality gives:

$$X_{\mu;\nu\lambda} g^{\nu\lambda} = X_{\sigma} R^{\sigma\nu}{}_{\nu\mu} = -X_{\sigma} R^{\sigma}{}_{\mu} \,. \tag{A.3}$$

So, if the Ricci tensor vanishes than any covector Killing field X_{μ} is a harmonic one-form:

$$X_{\mu}{}^{;\nu}{}_{\nu} = -X_{\sigma} R^{\sigma}{}_{\mu} = 0 \,.$$

A.2 Conformal Killing vectors

A natural conformal generalization of the equation (A.1) has the following form:

$$Z_{\lambda;\mu} + Z_{\mu;\lambda} - \frac{2}{n}g_{\lambda\mu}Z^{\sigma}_{\;;\sigma} = 0\,.$$

The solution Z^{λ} of this equation we call *conformal Killing vector field*. Let us denote $Z := Z^{\sigma}_{\;;\sigma}$. We can perform the similar trick as for Killing vectors, namely:

$$Z_{\lambda;\mu\nu} + Z_{\mu;\lambda\nu} - Z_{\lambda;\nu\mu} - Z_{\nu;\lambda\mu} + Z_{\nu;\mu\lambda} + Z_{\mu;\nu\lambda} - 2Z_{\mu;\nu\lambda} = -2Z_{\sigma}R^{\sigma}_{\;\lambda\nu\mu}\,,$$

and we obtain the following expression for second derivatives:

$$Z_{\mu;\nu\lambda} + \frac{1}{n}\left(g_{\lambda\nu}Z_{;\mu} - g_{\lambda\mu}Z_{;\nu} - g_{\mu\nu}Z_{;\lambda}\right) = Z_{\sigma}R^{\sigma}_{\;\lambda\nu\mu}\,. \tag{A.4}$$

Taking the trace in the indices $\mu\lambda$ we get

$$Z^{\mu}_{\;;\nu\mu} - Z_{;\nu} = Z_{\sigma}R^{\sigma}_{\;\nu}$$

which is trivial if we remember that $Z = Z^{\sigma}_{\;;\sigma}$.

The trace with respect to the indices $\mu\nu$ in equation (A.4) gives the identity, but the trace with respect to $\nu\lambda$ leads to the result:

$$Z_{\mu}^{\;;\nu}_{\;\;\nu} + \frac{n-2}{n}Z_{;\mu} = -Z_{\sigma}R^{\sigma}_{\;\mu}\,. \tag{A.5}$$

If we take the derivative with respect to the index μ (a contraction) in equation (A.4) and perform some further straightforward manipulations, we obtain the following result:

$$Z_{;\lambda\nu} + 2Z^{\sigma}_{\;;(\lambda}R_{\nu)\sigma} + Z^{\sigma}R_{\lambda\nu;\sigma} + \frac{1}{n}\left(g_{\lambda\nu}Z^{;\mu}_{\;\;\mu} - Z_{;\nu\lambda} - Z_{;\lambda\nu}\right) = 0\,. \tag{A.6}$$

We can also take a trace in (A.6):

$$Z^{;\mu}_{\;\;\mu} + \frac{n}{2(n-1)}\left(\frac{2}{n}ZR + Z^{\sigma}R_{;\sigma}\right) = 0\,,$$

where we used the following identity for the conformal vector field:

$$ZR = nZ_{\mu;\nu}R^{\mu\nu}\,.$$

And finally the second derivatives of Z fulfill the property:

$$\frac{n-2}{n}Z_{;\lambda\nu} - \frac{1}{n-1}g_{\lambda\nu}\left(\frac{1}{n}ZR + \frac{1}{2}Z^{\sigma}R_{;\sigma}\right) + Z^{\sigma}R_{\lambda\nu;\sigma} + 2Z^{\sigma}_{\;;(\lambda}R_{\nu)\sigma} = 0\,.$$

A.3 CYK tensors

$$f_{\lambda\mu;\nu} + f_{\nu\mu;\lambda} = \frac{2}{n-1}\left(g_{\nu\lambda}f^{\sigma}_{\;\mu;\sigma} + g_{\mu(\lambda}f_{\nu)}^{\;\;\sigma}_{\;;\sigma}\right) \tag{A.7}$$

According to the Definition 2 the skew-symmetric tensor $f_{\mu\nu}$ fulfilling the equation (A.7) we call the *conformal Yano–Killing tensor*.

Let us denote $f_{\kappa} := f^{\sigma}_{\;\kappa;\sigma}$. By the similar trick as for (A.2) and (A.4) we get:

$$2f_{\lambda\kappa;\nu\mu} = \frac{2}{n-1}\left(g_{\lambda\mu}f_{\kappa;\nu} + g_{\nu\lambda}f_{\kappa;\mu} - g_{\mu\nu}f_{\kappa;\lambda} - g_{\kappa(\lambda}f_{\mu);\nu} + g_{\kappa(\mu}f_{\nu);\lambda} - g_{\kappa(\nu}f_{\lambda);\mu}\right) +$$

18

$$+ f_{\sigma\lambda}R^{\sigma}{}_{\kappa\mu\nu} + f_{\sigma\mu}R^{\sigma}{}_{\kappa\lambda\nu} + f_{\sigma\nu}R^{\sigma}{}_{\kappa\lambda\mu} + 2f_{\sigma\kappa}R^{\sigma}{}_{\mu\nu\lambda}\,. \tag{A.8}$$

The trace in $\lambda\nu$ in equation (A.8) leads to the identity but the trace in $\kappa\nu$ gives the following result:

$$g_{\mu\nu}f^{\lambda}{}_{;\lambda} + (n-2)f_{(\mu;\nu)} = (n-1)R_{\sigma(\mu}f_{\nu)}{}^{\sigma}\,. \tag{A.9}$$

The trace of the equation (A.9) gives the equation:

$$f^{\lambda}{}_{;\lambda} = -\frac{1}{2}f^{\mu\nu}R_{\mu\nu} = 0\,.$$

<u>Remark:</u> If $R_{\mu\nu} = 0$ then from equation (A.9) $f_{(\mu;\nu)} = 0$, so f_{μ} is a Killing field.

The trace with respect to the indices $\mu\nu$ in equation (A.8) gives as follows:

$$g_{\mu\nu}f^{\lambda}{}_{;\lambda} + (n-2)f_{(\mu;\nu)} = (n-1)R_{\lambda(\mu\nu)\sigma}f^{\lambda\sigma} + (n-1)R_{\sigma(\mu}f_{\nu)}{}^{\sigma}\,, \tag{A.10}$$

$$f_{\lambda\kappa}{}^{;\mu}{}_{\mu} = f_{\sigma}{}^{\mu}R^{\sigma}{}_{[\kappa\lambda]\mu} + \frac{n-4}{n-1}f_{[\lambda;\kappa]} - f^{\sigma}{}_{[\kappa}R_{\lambda]\sigma}\,, \tag{A.11}$$

where we have written the symmetric and skew-symmetric parts separately. The equations (A.9) and (A.10) are equivalent because $R^{[\lambda}{}_{(\mu\nu)}{}^{\sigma]} = 0$, so $R_{\lambda(\mu\nu)\sigma}f^{\lambda\sigma} = 0$.

Let us notice that for $n = 4$ the second term on the right-hand side of the equation (A.11) vanishes and the final form of this equality is the following:

$$f_{\lambda\kappa}{}^{;\mu}{}_{\mu} = \frac{1}{2}f_{\sigma\mu}W^{\sigma\mu}{}_{\lambda\kappa} - \frac{1}{6}Rf_{\lambda\kappa}\,, \tag{A.12}$$

where by $W_{\lambda\mu\nu\kappa}$ we have denoted the Weyl tensor for the metric g. One can easily show that the equation A.12 is invariant with respect to the conformal rescaling i.e. f is a solution of $(\Box_g + \frac{1}{6}R)f = \frac{1}{2}Wf$ iff $\tilde{f} = \Omega^3 f$ is a solution of $(\Box_{\tilde{g}} + \frac{1}{6}\tilde{R})\tilde{f} = \frac{1}{2}\tilde{W}\tilde{f}$, where all objects $\Box_{\tilde{g}}, \tilde{R}, \tilde{W}$ are calculated with respect to the metric $\tilde{g} = \Omega^2 g$.

B   Functionals Θ on initial spacelike surface

B.1   Unconstrained degrees of freedom on Σ_t

In [18] it was shown that a pair of solutions $(\mathbf{x}, \mathbf{y})$ of the wave equation[7] gives a Weyl field in the following 2+2-form:

$$W_{abcd} = -\frac{1}{2}\rho^2\mathbf{x}\varepsilon_{ab}\varepsilon_{cd}$$

$$W_{ABcd} = -\frac{1}{2}\rho^2\mathbf{y}\varepsilon_{AB}\varepsilon_{cd}$$

$$W_a{}^B{}_{cd||B} = -\frac{1}{2}\varepsilon_{cd}\varepsilon^b{}_a\rho^3(\rho^{-1}\mathbf{x})_{,b}$$

$$W_{aBcd||E}\varepsilon^{BE} = -\frac{1}{2}\varepsilon_{cd}\rho^3(\rho^{-1}\mathbf{y})_{,a} \tag{B.1}$$

---

[7]The "mono-dipole" part of the field has a special form related to the Poincaré charges cf. [17] and [18].

$$\overset{\circ}{W}{}_c{}^{AB}{}_{d||AB} = \frac{1}{4}\rho^4 \mathbf{x}_{cd} = \frac{1}{2}\rho^4\left[(\rho^{-2}\mathbf{x})_{,cd} - \frac{1}{2}\eta_{cd}(\rho^{-2}\mathbf{x})^{,b}{}_b\right]$$

$$\overset{\circ}{W}{}_c{}^A{}_{Bd||AC}\varepsilon^{BC} = \frac{1}{4}\rho^4\left[\varepsilon_c{}^b(\rho^{-2}\mathbf{y})_{,bd} + \varepsilon_d{}^b(\rho^{-2}\mathbf{y})_{,bc}\right]$$

where indices A, B, C are related to the coordinates (θ, φ) along spheres

$$S(r) := \{t = \mathrm{const}, \ r = \mathrm{const}\}$$

but the indices a, b, c correspond to normal to $S(r)$ null coordinates $u = t - r$, $v = t + r$.

B.2   Constraints for E and H

We have the following non-vanishing Christoffel symbols for the three-metric η_{kl} on Σ_t in spherical coordinates $\Gamma^3{}_{AB} = -\frac{1}{r}\eta_{AB}$, $\Gamma^A{}_{3B} = \frac{1}{r}\delta^A{}_B$, and the spherical part $\Gamma^C{}_{AB}$ which is not dependent on radial coordinate $x^3 := r$ and defines covariant derivative on a sphere $r =$const which we denote by "$||$". The angular coordinates x^A parameterize spheres $S(r)$. Let us also denote by $\boldsymbol{\Delta}$ a Laplace–Beltrami operator on a unit sphere $S(1)$.

The 2+1-splitting of the constraint 5.22:

$$\partial_3(r^3 E^{33}) + r^3 E^{3A}{}_{||A} = 0, \tag{B.2}$$

$$\partial_3(r^4 E^{3A}{}_{||A}) + r^4 \overset{\circ}{E}{}^{AB}{}_{||AB} - \frac{1}{2}r^2\boldsymbol{\Delta}E^{33} = 0, \tag{B.3}$$

$$\partial_3(r^4 E^3{}_{A||B}\varepsilon^{AB}) + r^4 \overset{\circ}{E}{}_A{}^B{}_{||BC}\varepsilon^{AC} = 0 \tag{B.4}$$

allows us to express explicitly all electric components of the Weyl tensor in terms of $\mathbf{x}$ and $\partial_0\mathbf{y}$:

$$2r^2 E^{33} = \mathbf{x}, \quad 2r^2 E_{3A||B}\varepsilon^{AB} = -\partial_0\mathbf{y} \tag{B.5}$$

$$2r^3 E^{3A}{}_{||A} = -2\partial_3(r^3 E^{33}) = -\partial_3(r\mathbf{x}) \tag{B.6}$$

$$2r^4 \overset{\circ}{E}{}^{AB}{}_{||AB} = -2\partial_3(r^4 E^{3A}{}_{||A}) + \boldsymbol{\Delta}(r^2 E^{33}) = \partial_3\left(r\partial_3(r\mathbf{x})\right) + \frac{1}{2}\boldsymbol{\Delta}\mathbf{x} \tag{B.7}$$

$$2r^4 \overset{\circ}{E}{}_A{}^B{}_{||BC}\varepsilon^{AC} = -2\partial_3(r^4 E^3{}_{A||B}\varepsilon^{AB}) = \partial_3(r^2\partial_0\mathbf{y}) \tag{B.8}$$

Similarly, we get the magnetic part in terms of $\mathbf{y}$ and $\partial_0\mathbf{x}$:

$$2r^2 H^{33} = \mathbf{y} \tag{B.9}$$

$$2r^2 H_{3A||B}\varepsilon^{AB} = -\partial_0\mathbf{x} \tag{B.10}$$

$$2r^3 H^{3A}{}_{||A} = -\partial_3(r\mathbf{y}) \tag{B.11}$$

$$2r^4 \overset{\circ}{H}{}^{AB}{}_{||AB} = \partial_3\left(r\partial_3(r\mathbf{y})\right) + \frac{1}{2}\boldsymbol{\Delta}\mathbf{y} \tag{B.12}$$

$$2r^4 \overset{\circ}{H}{}_A{}^B{}_{||BC}\varepsilon^{AC} = \partial_3(r^2\partial_0\mathbf{x}) \tag{B.13}$$

20

B.3 Reduction of the quadratic forms and the functionals

The "spherical" method from Appendix B in [19] can be easily applied for the reduction of the functional $\int_V r^2(E^2 + H^2)$. Let us for simplicity restrict ourselves to the case of a three-dimensional ball $B(0, R)$ with radius R, $V = B(0, R) = S^2 \times [0, R]$, $\partial V = S(R)$, $\int_V = \int_0^R dr \int_{S(r)} d\theta d\varphi$. For exterior problems we also consider $V = \Sigma \setminus B(0, R)$.

The (2+1)-splitting of the tensor q_{kl} gives the following components on a sphere: two scalars on S^2 – $\overset{2}{q} := \eta^{AB} q_{AB}$ and q_{33}, vector q_{3A} on S^2 and symmetric traceless tensor $\overset{\circ}{q}_{AB} := q_{AB} - \frac{1}{2}\eta_{AB}\overset{2}{q}$. Let $p^{kl} := \sqrt{\det \eta_{ij}} q^{kl}$ be a tensor density on Σ. On each sphere $S(r)$ we can manipulate as follows

$$\int_{S(r)} p^{kl} q_{kl} = \int_{S(r)} p^{33} q_{33} + 2p^{3A} q_{3A} + \frac{1}{2}\overset{2}{p}\overset{2}{q} + \overset{\circ}{p}^{AB}\overset{\circ}{q}_{AB} =$$

$$= \int_{S(r)} p^{33} q_{33} - 2(rp^{3A}_{\ ||A})\boldsymbol{\Delta}^{-1}(rq_{3A}^{\ ||A}) - 2(rp^{3A||B}\varepsilon_{AB})\boldsymbol{\Delta}^{-1}(rq_{3A||B}\varepsilon^{AB}) + \frac{1}{2}\overset{2}{p}\overset{2}{q} +$$

$$+2\int_{S(r)} (r^2\varepsilon^{AC}\overset{\circ}{p}_A^{\ B}_{\ ||BC})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\overset{\circ}{q}_A^{\ B}_{\ ||BC})+$$

$$+2\int_{S(r)} (r^2\overset{\circ}{p}^{AB}_{\ ||AB})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r^2\overset{\circ}{q}^{AB}_{\ ||AB}),$$

where we have used the following identities on a sphere

$$-\int_{S(r)} \pi^A v_A = (r\pi^A_{\ ||A})\boldsymbol{\Delta}^{-1}(rv^A_{\ ||A}) + (r\pi^{A||B}\varepsilon_{AB})\boldsymbol{\Delta}^{-1}(rv_{A||B}\varepsilon^{AB}),\tag{B.14}$$

and similarly for the traceless tensors we have

$$\int_{S(r)} \overset{\circ}{\pi}^{AB}\overset{\circ}{v}_{AB} = 2\int_{S(r)} (r^2\varepsilon^{AC}\overset{\circ}{\pi}_A^{\ B}_{\ ||BC})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\overset{\circ}{v}_A^{\ B}_{\ ||BC})+$$

$$+2\int_{S(r)} (r^2\overset{\circ}{\pi}^{AB}_{\ ||AB})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r^2\overset{\circ}{v}^{AB}_{\ ||AB}).\tag{B.15}$$

The axial part of the quadratic form $\int_{S(r)} p^{kl} q_{kl}$ we define as

$$\text{axial part} = -2\int_{S(r)} (rp^{3A||B}\varepsilon_{AB})\boldsymbol{\Delta}^{-1}(rq_{3A||B}\varepsilon^{AB})+$$

$$+2\int_{S(r)} (r^2\varepsilon^{AC}\overset{\circ}{p}_A^{\ B}_{\ ||BC})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\overset{\circ}{q}_A^{\ B}_{\ ||BC}).$$

The remainder we define as a polar part. Using the above formulae by inserting into them $q_{kl} = H_{kl}$ and $q_{kl} = E_{kl}$ respectively we obtain the following expressions

$$\text{axial part of} \int_{S(r)} r^2 H^{kl} H_{kl} = \frac{1}{2}\int_{S(r)} (r\dot{\mathbf{x}})(-\boldsymbol{\Delta})^{-1}(r\dot{\mathbf{x}}) + \partial_r(r^2\dot{\mathbf{x}})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_r(r^2\dot{\mathbf{x}})$$

$$\text{polar part of} \int_{S(r)} r^2 E^{kl} E_{kl} = \frac{1}{2}\int_{S(r)} \frac{3}{2}\mathbf{x}^2 + \partial_r(r\mathbf{x})(-\boldsymbol{\Delta})^{-1}\partial_r(r\mathbf{x}) + \tag{B.16}$$

$$\left(\partial_r[r\partial_r(r\mathbf{x})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{x}\right)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\left(\partial_r[r\partial_r(r\mathbf{x})] + \frac{1}{2}\boldsymbol{\Delta}\mathbf{x}\right)$$

21

where we also used relations (B.5–B.8). Similarly from (B.9–B.13) we obtain

$$\text{polar} \int_{S(r)} r^2 E^{kl} E_{kl} + \text{axial} \int_{S(r)} r^2 H^{kl} H_{kl} = \frac{1}{2} \int_{S(r)} \frac{3}{2} \mathbf{y}^2 + \partial_r(r\mathbf{y})(-\mathbf{\Delta})^{-1}\partial_r(r\mathbf{y}) +$$

$$+\frac{1}{2} \int_{S(r)} \left(\partial_r[r\partial_r(r\mathbf{y})] + \frac{1}{2}\mathbf{\Delta}\mathbf{y}\right)\mathbf{\Delta}^{-1}(\mathbf{\Delta}+2)^{-1}\left(\partial_r[r\partial_r(r\mathbf{y})] + \frac{1}{2}\mathbf{\Delta}\mathbf{y}\right)$$

$$+\frac{1}{2} \int_{S(r)} (r\dot{\mathbf{y}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{y}}) + \partial_r(r^2\dot{\mathbf{y}})\mathbf{\Delta}^{-1}(\mathbf{\Delta}+2)^{-1}\partial_r(r^2\dot{\mathbf{y}}) \tag{B.17}$$

This way we obtain (5.25).

The similar manipulations give the following

$$\int_{S(r)} r^4 E_{kr} E^{kr} = \int_{S(r)} \left[(r^2 E_{rr})^2 - (r^3 E^{rA}{}_{||A})\mathbf{\Delta}^{-1}(r^3 E^{rA}{}_{||A}) \right.$$

$$\left. -(r^3 E_{rA||B}\varepsilon^{AB})\mathbf{\Delta}^{-1}(r^3 E_{rA||B}\varepsilon^{AB}) \right]$$

$$= \frac{1}{4} \int_{S(r)} \mathbf{x}^2 + \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}\partial_r(r\mathbf{x}) + (r\dot{\mathbf{y}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{y}})$$

and similarly

$$\int_{S(r)} r^4 H_{kr} H^{kr} = \frac{1}{4} \int_{S(r)} \mathbf{y}^2 + \partial_r(r\mathbf{y})(-\mathbf{\Delta})^{-1}\partial_r(r\mathbf{y}) + (r\dot{\mathbf{x}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}})$$

which finally gives (5.26).

It is also instructive to see how the conservation law for the functional (5.26) can be obtained straightforward from the wave equations

$$\Box\mathbf{x} = 0, \quad \Box\mathbf{y} = 0. \tag{B.18}$$

This can be shown as follows

$$\partial_0 \frac{1}{2} \int_V \left[\mathbf{x}^2 + \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}\partial_r(r\mathbf{x}) + (r\dot{\mathbf{x}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}})\right] \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi =$$

$$= \int_V \left[\mathbf{x}\dot{\mathbf{x}} + \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}\partial_r(r\dot{\mathbf{x}}) + (r\ddot{\mathbf{x}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}})\right] \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi =$$

$$= \int_V \left[\mathbf{\Delta}\mathbf{x} + r\partial_r(\partial_r(r\mathbf{x})) - r^2\ddot{\mathbf{x}}\right] \mathbf{\Delta}^{-1}\dot{\mathbf{x}}\mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$$

$$+ \int_{\partial V} \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}}) \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$$

$$= \int_V r^2 \Box\mathbf{x}\mathbf{\Delta}^{-1}\dot{\mathbf{x}}\mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi + \int_{\partial V} \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}}) \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$$

The volume term vanishes because of the wave equation (B.18) and to get the result

$$\partial_0 \int_V \left[\mathbf{x}^2 + \partial_r(r\mathbf{x})(-\mathbf{\Delta})^{-1}\partial_r(r\mathbf{x}) + (r\dot{\mathbf{x}})(-\mathbf{\Delta})^{-1}(r\dot{\mathbf{x}})\right] \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = 0 \tag{B.19}$$

we need to assume that $\mathbf{x}$ or rather $\dot{\mathbf{x}}$ vanishes on the boundary.

The other examples of our functionals Θ on surface Σ_0 can be expressed as follows:

$$
\begin{aligned}
\Theta(\mathcal{T}_0, V; \mathcal{T}_i \wedge \mathcal{T}_0) &= \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{T}_i \wedge \mathcal{T}_0)\big)\mathrm{d}^3 x \\
&= \frac{1}{2}\int_V \big(E_{ki}E^k{}_i + H_{ki}H^k{}_i\big)\, r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{1}{2}\int_V \Big[(vE^{kr} + rv_{||A}E^{kA})(vE_{kr} + rv_{||A}E_k{}^A) \\
&\quad + (vH^{kr} + rv_{||A}H^{kA})(vH_{kr} + rv_{||A}H_k{}^A)\Big] r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{1}{2}\int_V v^2\Big[E^{kl}E_{kl} + \frac{1}{2}r^2(E^{kA}E_k{}^B)_{||AB} - r(E^{kA}E_{kr})_{||A} \\
&\quad + H^{kl}H_{kl} + \frac{1}{2}r^2(H^{kA}H_k{}^B)_{||AB} - r(H^{kA}H_{kr})_{||A}\Big] r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,,
\end{aligned}
\tag{B.20}
$$

where $v = \frac{x_i}{r}$,

$$
\begin{aligned}
\Theta(\mathcal{T}_0, V; \mathcal{T}_i \wedge \mathcal{D}) &= \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{T}_i \wedge \mathcal{D})\big)\mathrm{d}^3 x \\
&= \frac{1}{2}\int_V v_{||A}v_{||C}\varepsilon^{AB}\varepsilon^{CD}\Big(E^{kB}E_k{}^D + H^{kB}H_k{}^D\Big)r^6\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{1}{2}\int_V v^2\Big[E^{kA}E_{kA} + H^{kA}H_{kA} \\
&\quad - \frac{1}{2}r^2\varepsilon^{AB}\varepsilon^{CD}(E_{kB}E^k{}_D + H_{kB}H^k{}_D)_{||AC}\Big]r^4\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,,
\end{aligned}
\tag{B.21}
$$

and for $Q_k^{\mathrm{boost}} := \mathcal{D} \wedge \mathcal{L}_{0k} - \frac{1}{2}x^\mu x_\mu \mathcal{T}_0 \wedge \mathcal{T}_k$ we get

$$
\begin{aligned}
\Theta(\mathcal{T}_0, V; Q_i^{\mathrm{boost}}) &= \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, Q_i^{\mathrm{boost}})\big)\mathrm{d}^3 x \\
&= \frac{1}{2}\int_V r^4\Big[(vE^{kr} - rv_{||A}E^{kA})(vE_{kr} - rv_{||B}E_k{}^B) \\
&\quad + (vH^{kr} - rv_{||A}H^{kA})(vH_{kr} - rv_{||B}H_k{}^B)\Big]r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{1}{2}\int_V v^2\Big[E^{kl}E_{kl} + \frac{1}{2}r^2(E^{kA}E_k{}^B)_{||AB} + r(E^{kA}E_{kr})_{||A} \\
&\quad + H^{kl}H_{kl} + \frac{1}{2}r^2(H^{kA}H_k{}^B)_{||AB} + r(H^{kA}H_{kr})_{||A}\Big] r^6\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,.
\end{aligned}
\tag{B.22}
$$

Hence, from the identity $\sum_{i=1}^3 \left(\frac{x_i}{r}\right)^2 = 1$ one can easily check that

$$
2\sum_{k=1}^3 \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{T}_k \wedge \mathcal{T}_0)\big)\mathrm{d}^3 x = \int_V T^{BR}\big(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\big)\mathrm{d}^3 x
\tag{B.23}
$$

$$
2\sum_{k=1}^3 \int_V T^{EM}\big(\mathcal{K}_0, \mathcal{T}_0, F(W, \mathcal{T}_k \wedge \mathcal{T}_0)\big)\mathrm{d}^3 x = \int_V T^{BR}\big(\mathcal{K}_0, \mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\big)\mathrm{d}^3 x
\tag{B.24}
$$

$$
2\sum_{\mu=0}^3 \int_V T^{EM}\big(\mathcal{T}_0, \mathcal{T}_0, F(W, \mathcal{T}_\mu \wedge \mathcal{D})\big)\mathrm{d}^3 x = \int_V T^{BR}\big(\mathcal{K}_0, \mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0\big)\mathrm{d}^3 x
\tag{B.25}
$$

23

$$2\sum_{\mu=0}^{3}\int_V T^{EM}\big(\mathcal{K}_0,\mathcal{T}_0,F(W,\mathcal{T}_\mu\wedge\mathcal{D})\big)\mathrm{d}^3x=\int_V T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0\big)\mathrm{d}^3x \qquad\text{(B.26)}$$

$$2\sum_{k=1}^{3}\int_V T^{EM}\big(\mathcal{T}_0,\mathcal{T}_0,F(W,Q_k^{\mathrm{boost}})\big)\mathrm{d}^3x=\int_V T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0,\mathcal{T}_0\big)\mathrm{d}^3x\,. \qquad\text{(B.27)}$$

$$2\sum_{k=1}^{3}\int_V T^{EM}\big(\mathcal{K}_0,\mathcal{T}_0,F(W,Q_k^{\mathrm{boost}})\big)\mathrm{d}^3x=\int_V T^{BR}\big(\mathcal{K}_0,\mathcal{K}_0,\mathcal{K}_0,\mathcal{T}_0\big)\mathrm{d}^3x\,. \qquad\text{(B.28)}$$

The above formulae show explicitly how the functionals Θ include the functionals CQ^{BR}.

C   Index of symbols

M – spacetime

$;\mu$ – four-dim. covariant derivative also denoted by ∇_μ

$\varepsilon_{\mu\nu\gamma\delta}$ – Levi–Civita skew-symmetric tensor in spacetime M

$*$ – Hodge dual operation, $t^*_{\mu\nu}:=\tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\kappa}t_{\lambda\kappa}$

∂_μ – partial derivative also denoted by "," (comma)

$\eta_{\mu\nu}$ – Minkowski metric

x^μ – Cartesian coordinates in Minkowski spacetime

$\mathcal{T}_\mu$ – translation Killing field, $\mathcal{T}_\mu:=\frac{\partial}{\partial x^\mu}$

$\mathcal{L}_{\mu\nu}$ – boost or rotation Killing field, $\mathcal{L}_{\mu\nu}:=x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}$

$\mathcal{D}$ – scaling conformal Killing field, $\mathcal{D}:=x^\nu\frac{\partial}{\partial x^\nu}$

$\mathcal{K}_\mu$ – "acceleration" conformal Killing field, $\mathcal{K}_\mu:=-2x_\mu\mathcal{D}+x^\sigma x_\sigma\mathcal{T}_\mu$

t – time coordinate

r – radial coordinate

x – "inverse" radial coordinate $r=x^{-1}$

u,v – null coordinates on M, $u=t-r$, $v=t+r$

θ,ϕ – spherical coordinates on S^2

d – exterior derivative

$\mu,\nu,\ldots$ – four-dimensional indices running $0,\ldots,3$

$k,l,\ldots$ – three-dimensional indices running $1,\ldots,3$

$A,B,\ldots$ – two-dimensional indices on a sphere

$\Box$ – d'Alambertian, wave operator

R – scalar curvature

X – vector field

i^0 – spatial infinity

$\mathscr{I}$ – null infinity

$\mathscr{I}^+$ – future null infinity

$\mathscr{I}^-$ – past null infinity

$\delta^\mu{}_\nu$ – Kronecker's delta

$S(r)$ – sphere with radius r parameterized by θ, ϕ

η_{AB} – metric on $S(r)$

$\mathring{t}_{AB}$ – two-dim. traceless part of the tensor t_{AB}, $\mathring{t}_{AB} := t_{AB} - \frac{1}{2}\eta_{AB}\eta^{CD}t_{CD}$

$\|$ – two-dimensional covariant derivative on a sphere

$\mathbf{\Delta}$ – two-dimensional laplacian on a unit sphere

ε^{AB} – two-dimensional skew-symmetric tensor on $S(r)$, $r^2 \sin\theta \varepsilon^{\theta\phi} = 1$

ε^{ab} – skew-symmetric tensor on the two-dimensional space orthogonal to $S(r)$, $\varepsilon^{uv} = 2$

$R^\mu{}_{\nu\lambda\sigma}$ – curvature tensor

$R_{\mu\nu}$ – Ricci tensor

$\Gamma^\lambda{}_{\mu\nu}$ – Christoffel symbol

$W_{\mu\nu\lambda\sigma}$ – spin-2 field, Weyl tensor

$\mathbf{x}, \mathbf{y}$ – gauge-invariants

p_μ – four-momentum charge

b_μ – dual four-momentum charge

$j_{\mu\nu}$ – angular momentum and static moment charges

$w_{\mu\nu}$ – "ofam" charge

$Q_{\mu\nu}$ – CYK tensor

$\mathcal{Q}_{\mu\nu\lambda}$ – CYK equation

Ω – conformal factor

$F_{\mu\nu}$ – Maxwell field

$T^{EM}_{\mu\nu}$ – energy-momentum tensor for Maxwell field

$T^{BR}_{\mu\nu\alpha\beta}$ – Bel–Robinson tensor

CQ^{EM} – conserved quantity for Maxwell field

CQ^{BR} – conserved quantity for Bel–Robinson tensor

Θ – conserved quantity for CYK tensor bilinear in Weyl field

Σ – spacelike hypersurface

E – electric part of Weyl field

H – magnetic part of Weyl field

$T_{\mu\nu\alpha\beta\gamma\delta}$ – (super-)tensor

References

[1] D. Baleanu, Nuovo Cimento B 114, 1065–1072 (1999)

[2] I.M. Benn, P. Charlton and J. Kress, Journal of Mathematical Physics 38, 4504–4527 (1997)

[3] S. Benenti, M. Francaviglia, General Relativity and Gravitation, Edited by A. Held, vol. 1, 393–439, Plenum Press, (New York 1980)

[4] I. Białynicki–Birula and Z. Białynicka–Birula, *Quantum Electrodynamics*, Pergamon, (Oxford 1975)

[5] D. Christodoulou, S. Klainerman, Communications on Pure and Applied Mathematics 43, 137–199 (1990)

[6] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, Princeton University Press (1993)

[7] P.T. Chruściel and E. Delay, *Existence of non-trivial, vacuum, asymptotically simple space-times*, gr-qc/0203053

[8] P.T. Chruściel, J. Jezierski and J. Kijowski, *Hamiltonian Field Theory in the Radiating Regime*, Lecture Notes in Physics: m70, Springer (2002)

[9] C.D. Collinson and L. Howarth, General Relativity and Gravitation 32, 1767–1776 (2000); General Relativity and Gravitation 32, 1845–1849 (2000)

[10] J. Corvino, Commun. Math. Phys. 214, 137–189 (2000); J. Corvino and R. Schoen, *Vacuum spacetimes which are identically Schwarzschild near spatial infinity*, talk given at the Santa Barbara Conference on Strong Gravitational Fields, June 22-26 (1999), http://doug-pc.itp.ucsb.edu/online/gravity_c99/schoen/

[11] W. Dietz and R. Rudiger, Proc. Roy. Soc. Lond. A Mat. 37, 361–378 (1981)

[12] H. Friedrich, *Asymptotic Structure of Space-Time*, in: Recent Advances in General Relativity, Edited by A.I. Janis and J.R. Porter, Birkhäuser, (Basel 1992)

[13] T. Friedrich, *Self–duality of Riemannian Manifolds and Connections*, in: Self–dual Riemannian Geometry and Instantons, TEUBNER–TEXTE zur Mathematik, Band 34, (Leipzig 1981)

[14] G.W. Gibbons, P.J. Ruback, Commun. Math. Phys. 115, 267–300 (1988)

[15] J.N. Goldberg, Physical Review D 41, 410–416 (1990)

[16] N.G. Ibohal, Astrophysics and Space Science 249, 73–93 (1997)

[17] J. Jezierski, General Relativity and Gravitation 27, 821–843 (1995)

[18] J. Jezierski, Classical and Quantum Gravity 14, 1679–1688 (1997)

[19] J. Jezierski, *Gauge-invariant formulation of the linearized Einstein equations on the Schwarzschild background*, in Current Topics in Math. Cosmology, eds M.Rainer and H-J Schmidt, World Scientific (1998); General Relativity and Gravitation 31, 1855-1890 (1999), (gr-qc/9801068)

[20] J. Jezierski, Classical and Quantum Gravity 19, 2463–2490 (2002)

[21] R. Penrose and W. Rindler, *Spinors and Space-time*, Cambridge University Press, Vol. 2, p.396 (Cambridge 1986)

[22] R.H. Rietdijk, Classical and Quantum Gravity 7, 247 (1990)

[23] J.M.M. Senovilla, Classical and Quantum Gravity 17, 2799–2842 (2000)

[24] J. Tafel and S. Pukas, Classical and Quantum Gravity 17, 1559–1570 (2000); J. Tafel, Classical and Quantum Gravity 17, 4397–4408 (2000)

[25] J.W. van Holten, Phys. Lett. B 342, 47 (1995)

[26] D. Vaman and M. Visinescu, Phys. Rev. D 54, 1398–1402 (1996)

[27] K. Yano, Ann. Math. 55, 328 (1952)